Admissibility Toolbox

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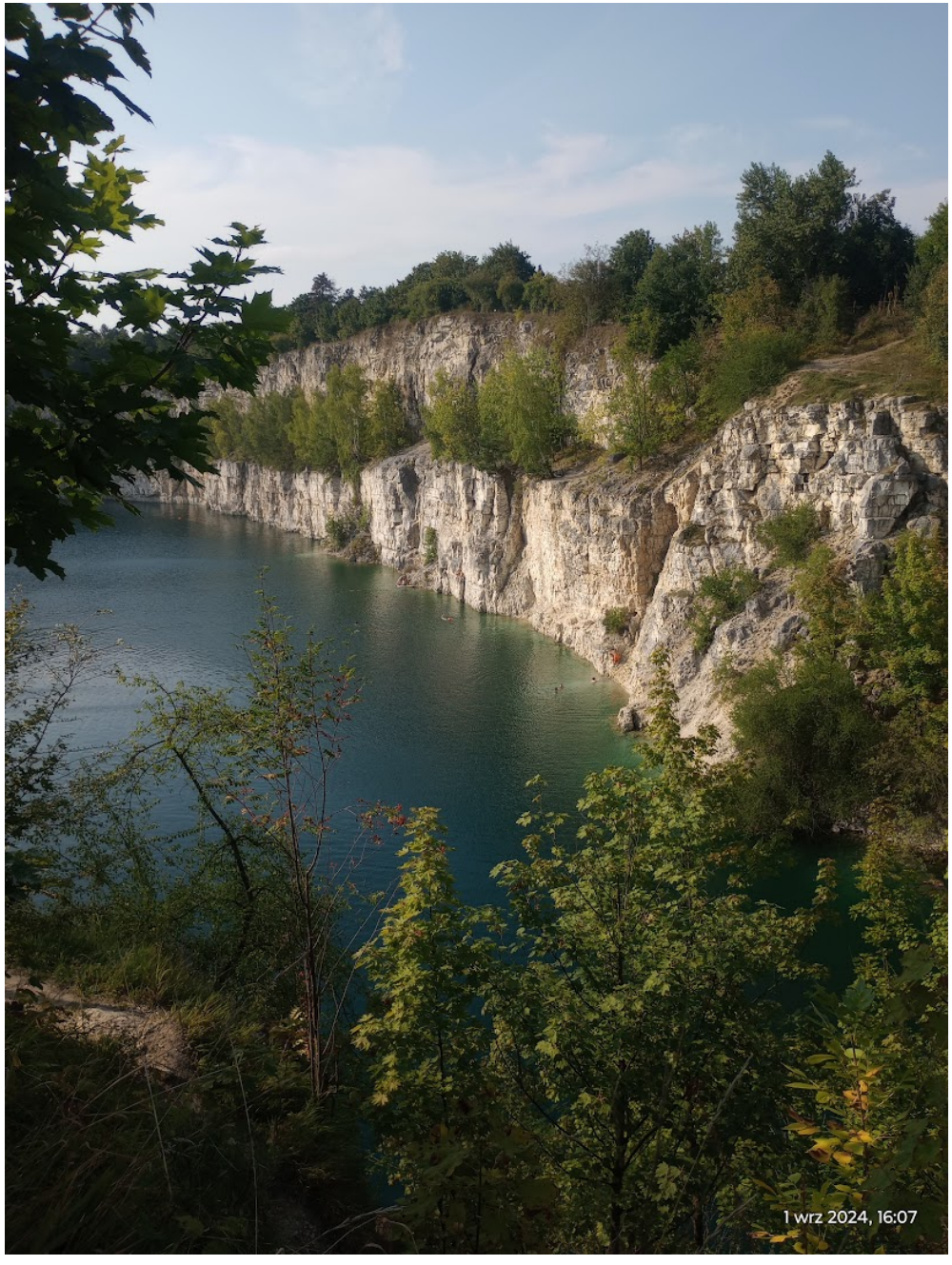
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Introduction

Definition

Let L be a logic. Then a rule $\frac{\Delta}{\gamma}$ is **admissible** if for every substitution s we have

$$s(\Delta) \subseteq L \quad \text{yields} \quad s(\gamma) \in L.$$

Let us consider a rule $R : \frac{\Box x}{x}$.

Fact

The rule R is not derivable in modal logic K. Indeed, it fails in $\circ$, the irreflexive singleton frame.

Fact

But R is admissible in K.

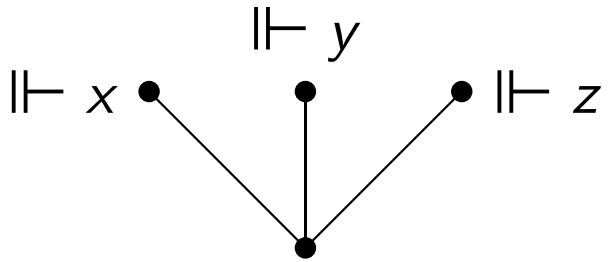
For suppose that $s(x) = \varphi$ is a substitution such that $\varphi \notin \mathrm{K}$. By Kripke completeness, there is a Kripke model $\mathbb{M} = (\mathbb{F}, \mathrm{val})$ and a vertex w such that $w \nVdash \varphi$. Let $\mathbb{M}'$ be the model obtained from $\mathbb{M}$ by adding one vertex w' and the relation $w' R w$. Then $w' \nVdash \Box\varphi$ in $\mathbb{M}'$.

Let us consider Kreisel-Putnam rule KP : $\dfrac{\neg x \to (y \vee z)}{(\neg x \to y) \vee (\neg x \to z)}$.

Fact

The rule KP is not derivable in intuitionistic logic INT.

Indeed, it fails in the depicted model.



Example

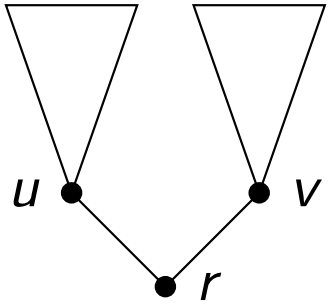
Fact

But *KP* is admissible in INT.

For suppose that $s(x) = \varphi$, $s(y) = \psi$, $s(z) = \chi$ be a substitution such that t $(\neg\varphi \to \psi) \vee (\neg\varphi \to \chi) \notin$ INT.
By Kripke completeness, there is a Kripke model $\mathbb{M} = (\mathbb{F}, \mathrm{val})$ (based on poset frame) and a vertex w such that $w \Vdash\!\!\!\!\!\!\vdash (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$. Then there are $u, v \geqslant w$ such that

$$u \Vdash \neg\varphi, \quad u \Vdash\!\!\!\!\!\!\vdash \neg\psi, \quad v \Vdash \neg\varphi, \quad v \Vdash\!\!\!\!\!\!\vdash \neg\chi.$$

Let $\mathbb{M}'$ be the model with the set of vertices $\uparrow u \uplus \uparrow v \uplus \{r\}$, with the order as on the picture and with a valuation val' that agrees with val on $\uparrow u$ and $\uparrow v$. Then



$r \Vdash \neg\varphi$ and $r \Vdash\!\!\!\!\!\!\vdash \varphi \vee \chi$,
and $r \Vdash\!\!\!\!\!\!\vdash \neg\varphi \to (\varphi \vee \chi)$.

Example

Let us consider the first Visser rule
$V_1 : \dfrac{(x\to y)\to(z\vee t)}{((x\to y)\to x)\vee((x\to y)\to z)\vee((x\to y)\to t)}$.

Fact

The rule V_1 is admissible INT.

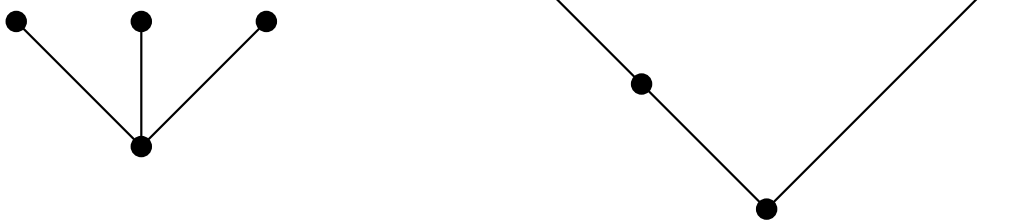
Proof. Let $s(x) = \varphi, s(y) = \psi, s(z) = \chi, s(t) = \tau$ be a substitution putting the conclusion outside INT. Let $\mathbb{M} = (\mathbb{F}, \mathrm{val})$ be a counter model. Then there are vertices u, v in $\mathbb{M}$ such that

$$u \Vdash \varphi \to \psi, \quad u \nVdash \varphi,$$
$$v \Vdash \varphi \to \psi, \quad v \nVdash \chi, \quad w \Vdash \varphi \to \psi, \quad w \nVdash \tau.$$

Define $\mathbb{F}' = {\uparrow}u \cup {\uparrow}v \cup {\uparrow}w \cup \{r\}$. Let val' be any extension of val on $\mathbb{F}'$. Suppose that $r \nVdash \varphi \to \psi$. Then $t \Vdash \varphi$, $t \nVdash \psi$ for some $t \geqslant r$. The only option for t is r. But it contradicts the satisfaction in u. Thus $r \Vdash \varphi \to \psi$, and $(\mathbb{F}, \mathrm{val}')$ is a countermodel for $(\varphi \to \psi) \to (\chi \vee \tau)$.

Exercise

Show that V_1 fails in the following Kripke frames.



Example

Exercise

Show that $r : \frac{\Diamond x, \Diamond \neg x}{0}$ is admissible but not derivable in modal logic S4 (or S5).

Definition (in other words)

A (structural) rule R is **admissible** in a logic L if, whenever applied to a theorem, it returns a theorem.

Or, when L is an *invariant* of R.

Perspective

Studying admissibility for a logic L is studying its transformations.

One may compare it to studying auto/end/poli/homeo-morphisms of mathematical structures.

And, as such, their understanding is the understanding of a logic itself.

When all admissible rules are derivable, or at least describable in some sense, we get better insight into a logic.

Algebraically, admissible rules correspond to the quasi-equational or, more generally, to universal theory of free algebras. These are key objects in algebra.

From a proof theory perspective, admissible rules act as **shortcuts** in derivations (the most prominent example being the CUT rule).

However, the computational aspect of these shortcuts, and the complexity of eliminating them, is hardly touched in current research.

The 1975 Impetus

The systematic study of admissibility in propositional non-classical logics was brought into sharp focus by **Harvey Friedman's** famous list of open problems (1975).

Two Complementary Problems

Friedman actually posed **two distinct problems** regarding admissibility:

- **Problem 41:** Demanded an algorithm to decide whether a rule is admissible in IPC (solved affirmatively by Rybakov, 1984).
- **Problem 42:** Concerned structural completeness and the existence of intermediate logics with the disjunction property (solved by Prucnal, 1979).

Aim of this mini course

Aim

- Introduce and invite to admissibility business.
- Present semantical approach, this is, algebraic and relational tools in studding admissibility.
- I particular, present sample arguments.

Apology

The course does not contain results concerning admissibility

- in non-Hilbert calculi, like sequent proof systems;
- in first order logics and their fragments;
- in many important non-classical logics (fuzzy logic, relevant logic, non-algebraizable).

However, there are many great results concerning admissibility in these areas. Many researchers work there.

I apologize for all omissions.

‘

Remark

Technical and/or demanding, even if important, proofs will be also omitted.

Toolbox

A **language** $\mathcal{L}$ is a set of connectives with their arities.
Let X be an infinite set of (propositional) variables, $Form(X)$ a the set of (propositional formulas) in a language $\mathcal{L}$.

Definition

A **substitution** is a mapping $s\colon X \to Form$, or its recursive extension $s\colon Form(x) \to Form(X)$.

Definition

A **rule** is a pair (Γ, δ), where Γ is a finite set of formulas and δ is a formula.
Rules are denoted by Γ/δ or $\frac{\Gamma}{\delta}$ or, when $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$, by $\varphi_1, \ldots, \varphi_n/\delta$ and $\frac{\varphi_1, \cdots, \varphi_n}{\delta}$.
Γ is the **premise** and δ is the **conclusion** of Γ/δ.

Definition

A (finitary, structural) consequence relation $\vdash$ is a set of rules satisfying the following postulates:

(Ref) If $\varphi \in \Gamma$, then $\Gamma \vdash \varphi$.

(Mon) If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$.

(Cut) If $\Gamma \vdash \delta$, for every $\delta \in \Delta$, and $\Delta \vdash \varphi$, then $\Gamma \vdash \varphi$.

(Sub) If $\Gamma \vdash \varphi$, then $\sigma(\Gamma) \vdash \sigma(\varphi)$.

Here $\Gamma \vdash \delta$ simply means that $\Gamma/\delta \in \vdash$.

Definition

A **deductive system** is a pair (Ax, Rl), where Ax is a set of formulas (called *axioms*) and Rl is a set of rules.
The division is not strict, as we may treat an axiom as a rule with empty premise. It, however, reflexes the way we think on axioms and rules.

Definition

A **proof** of φ in a deductive system is a pair (Ax, Rl) is a list of formulas, ending with φ, such that each of them

- is $s(\delta)$ for some substitution s and $\delta \in Ax$, or
- is of the form $s(\delta)$, where δ is the conclusion of a rule $\gamma_1, \ldots, \gamma_n/\delta \in Rl$ and $s(\gamma_1), \ldots, s(\gamma_n)$ appear earlier on the list.

Definition

A rule Γ / δ is derivable iff (Ax, Rl) if there is a proof of δ in $(Ax \cup \Gamma, Rl)$.

Definition

The set of derivable rules $\vdash$, for a deductive system (Ax, Rl) is a consequence relation.

Remark

Clearly, every consequence $\vdash$ relation is a consequence relation for some deductive system (e.g., for $(\varnothing, \vdash)$). This is why we say that a rule in derivable R in $\vdash$ if $R \in \vdash$.

Definition

A formula δ is a **theorem** for a deductive system (Ax, Rl), or its consequence relation $\vdash$, if $\vdash \delta$, this is, if it has a proof in (Ax, Rl).

Definition

A **logic** L is a set of formulas that

- It is closed under every substitution s,
  i.e., $s(\mathrm{L}) \subseteq L$;
- it contains a fixed set of axioms Ax;
- it is closed under a *default* set of rules RI,
  i.e., $s(\delta) \in \mathrm{L}$ whenever $\Gamma / \delta \in RI$ and $s(\Gamma) \subseteq \mathrm{L}$.

In other worlds, if L is the set of theorems for (Ax, RI).

Ax is then an **axiomatization**.
Actually, when we are not interested in axiomatization, we may treat the second condition implicit by putting $Ax = \mathrm{L}$.

Definition

An **extension** of a logic L is a logic with the same default rules as L and containing L.

Examples of logics

- CPL, the set of classical tautologies, is a logic with *modus ponens* as a default rule:

  $$MP : \frac{x, x \to y}{y}.$$

  CL has one proper extension, the inconsistent logic $Form(X)$.
- Intuitionistic logic INT with with MP as a default rule.
  Its consistent extensions are called **intermediate** logics.
- Modal logic K, axiomatized by $\Box(x \to y) \to (\Box x \to \Box y)$ with modus ponens and *necessitation* as default rules:

  $$N : \frac{x}{\Box x}.$$

  Among its (normal) extensions, let us distinguish
  S4, axiomatized by $\{\Box x \to x, \Box x \to \Box\,\Box\, x\}$, and
  S5, axiomatized by $\{\Diamond x \to \Box\,\Diamond\, x, \Box x \to \Box\,\Box\, x\}$
- Basic logic with MP as a default rule.
  Its extensions includes Łukasiewicz, Gödel, and product logics.

Intuitionistic logic INT: axiomatization

(In language with connectives: 0, $\wedge$, $\vee$, $\rightarrow$.)

Axioms:

(A1) $x \rightarrow (y \rightarrow x)$

(A2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))$

(A3-4) $x \wedge y \rightarrow x$, $x \wedge y \rightarrow y$

(A5) $x \rightarrow (y \rightarrow x \wedge y)$

(A6-7) $x \rightarrow x \vee y$, $y \rightarrow x \vee y$

(A8) $(x \rightarrow z) \rightarrow ((y \rightarrow z) \rightarrow (x \vee y \rightarrow z))$

(A9) $0 \rightarrow x$

A rule of inference: Modus Ponens.

By an intermediate logic we mean a set L of formulas such that

- $\mathrm{INT} \subseteq L \subseteq \mathrm{CL}$.
- L is closed under Modus Ponens:
  $x, x \to y \in L$ yields $y \in L$;
- L is closed under substitutions :
  for every substitution s, $\varphi \in L$ yields $s(\varphi) \in L$.

Examples:

- Gödel-Dummett logic LC, axiomatized relative to INT by $(x \to y) \vee (y \to x)$,
- Depth bounded by 2 logic BD2, axiomatized relative to INT by $x \vee (x \to (y \vee (y \to \bot)))$.

There are continuum many intermediate logics.

A **Kripke frame** (for INT) is a poset $\mathbb{F}$.
A **Kipke model** (for INT) is a Kripke frame equipped with a valuation

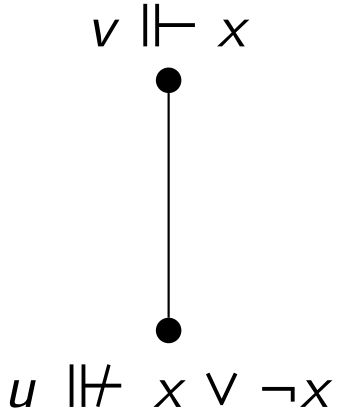
$$\mathrm{val}\colon X \to \mathrm{Upset}(\mathbb{F}).$$

Here $\mathrm{Upset}(\mathbb{F})$ is the set of up-sets (upward closed sets) in $\mathbb{F}$.

The valuation for formulas is obtained by recursion:
$\mathrm{val}(\bot) = \varnothing$, $\mathrm{val}(\varphi \wedge \psi) = \mathrm{val}(\varphi) \cap \mathrm{val}(\psi)$,
$\mathrm{val}(\varphi \vee \psi) = \mathrm{val}(\varphi) \cup \mathrm{val}(\psi)$,
$\mathrm{val}(\varphi \to \psi) = \{u : \forall v \text{ if } u \leqslant v \text{ and } v \in \mathrm{val}(\varphi) \text{ then } v \in \mathrm{val}(\psi)\}$,
$V(\neg\varphi) = \{u : \forall v \text{ if } u \leqslant v \text{ then } v \notin \mathrm{val}(\varphi)\}$.

We write $u \Vdash \varphi$ if $u \in \mathrm{val}(\varphi)$.

A Kripke model $(\mathbb{F}, \mathrm{val})$ satisfies φ if $\mathrm{val}(\varphi) = F$ (the carrier of $\mathbb{F}$),
A Kripke frame $\mathbb{F}$ satisfies φ ($\mathbb{F} \vDash \varphi$) provided that
$(\mathbb{F}, \mathrm{val})$ satisfies φ for every valuation val.
Here $\neg\varphi$ is $\varphi \to 0$.



$\mathrm{val}(\neg x) = \varnothing$
$\mathrm{val}(x \vee \neg x) = \{v\}$

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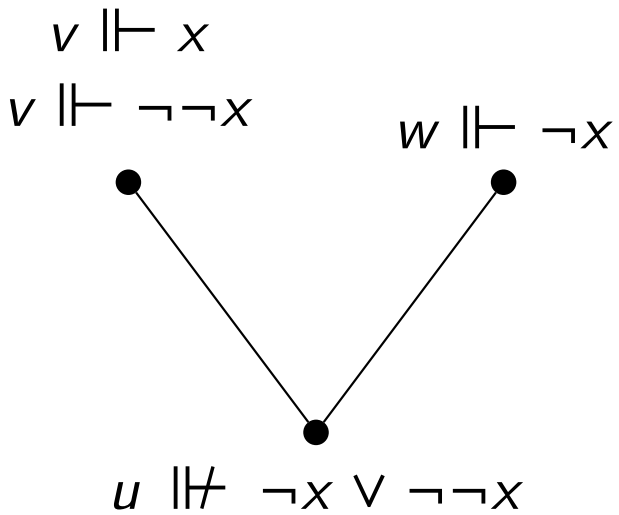


$\mathrm{val}(\neg x \vee \neg\neg x) = \{v, w\}$

Theorem (Kripke)

Logic INT consists of all formulas satisfied in all (finite) Kripke frames.

Theorem

Gödel-Dummett logic LC consists of all formulas satisfied in all (finite) Kripke frames which are linearly ordered posets.

Theorem

Bounded depth by 2 logic BD2 consists of all formulas satisfied in all (finite) Kripke frames which are posets of depth 2.

Fact

Logic LC $\oplus$ BD2 consists of all formulas valid in a two element chain.

Definition

- An intermediate logic, for which there is a set of frames characterizing it, is **Kriple complete**.
- An intermediate logic, for which there is a set of *finite* frames characterizing it, has the **finite model property**.
- An intermediate logic, for which there is a *finite* frame characterizing it, is **tabular**.

For us, also **locally tabular** logics will be relevant. But it is easier to define them algebraically.

Local tabularity yields finite model property.

LC and BD2 are locally tabular but not tabular.

Definition

A rule $\frac{\varphi_1,\ldots,\varphi_n}{\delta}$ is **valid** in a model $\mathbb{M} = (\mathbb{F}, \mathsf{val})$ iff

$$\mathbb{M} \vDash \varphi_1, \ldots \mathbb{M} \vDash \varphi_1 \quad \text{yields} \quad \mathbb{M} \vDash \delta.$$

A rule is **valid** in a frame $\mathbb{F}$ if it is valid in every $\mathbb{F}$-based model.

Language: 0, $\neg$, $\wedge$, $\vee$, $\Box$.
Additional connectives: $\varphi \to \psi = \neg\varphi \vee \psi$,
$\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$, $\Diamond\varphi = \neg \Box \neg\varphi$.

Axioms for the basic system K:

(A1) All classical propositional tautologies

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Rules of inference:
(Modus Ponens) if φ and $\varphi \to \psi$ then ψ
(Necessitation) if φ then $\Box\varphi$

Extensions of K, closed under MP and P, are called *normal*:

K4	(4)	$\Box\varphi \to \Box\Box\varphi$
S4	(4) and (T)	$\Box\varphi \to \varphi$
S5	(4), (T), and (B)	$\varphi \to \Box\Diamond\varphi$

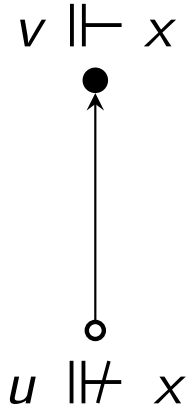
A **Kripke frame** (for modal logics) is a pair $\mathbb{F} = (F, R)$, $R \subseteq F^2$.
A **Kripke model** (for modal logics) is a Kripke frame equipped with a valuation

$$\mathrm{val}\colon X \to \mathcal{P}(F).$$

The valuation is extended to all formulas by recursion:
$\mathrm{val}(o) = \varnothing$, $\mathrm{val}(\neg\varphi) = F \setminus \mathrm{val}(\varphi)$, $\mathrm{val}(\varphi \wedge \psi) = \mathrm{val}(\varphi) \cap \mathrm{val}(\psi)$,
$\mathrm{val}(\varphi \vee \psi) = \mathrm{val}(\varphi) \cup \mathrm{val}(\psi)$, $\mathrm{val}(\varphi \to \psi) = (F \setminus \mathrm{val}(\varphi)) \cup \mathrm{val}(\psi)$,
$\mathrm{val}(\Box\varphi) = \{w : \forall v \text{ if } w\,R\,v \text{ then } v \in \mathrm{val}(\varphi)\}$,
$\mathrm{val}(\Diamond\varphi) = \{w : \exists v \text{ such that } w\,R\,v \text{ and } v \in \mathrm{val}(\varphi)\}$.

We write $w \Vdash \varphi$ if $w \in \mathrm{val}(\varphi)$.

A Kripke model $(\mathbb{F}, \mathrm{val})$ satisfies φ if $\mathrm{val}(\varphi) = F$,
A Kripke frame $\mathbb{F}$ satisfies φ ($\mathbb{F} \vDash \varphi$) provided that
$(\mathbb{F}, \mathrm{val})$ satisfies φ for every valuation val.



$\mathrm{val}(\Box x) = \{u, v\}$
$\mathrm{val}(\Diamond x) = \{u, v\}$

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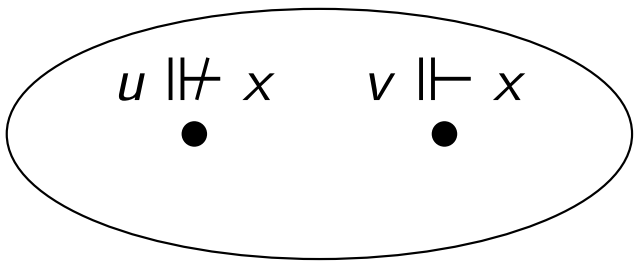
$\mathbb{C}_2$



$\mathrm{val}(\Box x) = \varnothing$
$\mathrm{val}(\Diamond x) = \{u, v\}$

Here $R = C_2^2$.

Theorem (Kripke)

Logic K consists of all formulas satisfied in all (finite) Kripke frames.

Theorem

Logic K4 consists of all formulas satisfied in all (finite) transitive Kripke frames.

Theorem

Logic S4 consists of all formulas satisfied in all (finite) reflexive and transitive Kripke frames (quasi-orders).

Theorem

Logic S5 consists of all formulas satisfied in all (finite) equivalence frames (reflexive, transitive, symmetric).

Definition

- A normal modal logic, for which there is a set of frames characterizing it, is **Kripke complete**.
- A normal modal logic, for which there is a set of *finite* frames characterizing it, has the **finite model property**.
- A normal modal logic, for which there is a *finite* frame characterizing it, is **tabular**.

For us, also **locally tabular** logics will be relevant. But it is easier to define them algebraically.
Local tabularity yields finite model property.

S5 is locally tabular but not tabular: it is complete with respect to the class of all finite equivalence frames, but no single finite frame suffices.

K, K4 and S4 are not locally tabular but have finite model property.

Definition

A rule $\frac{\varphi_1,\ldots,\varphi_n}{\delta}$ is **valid** in a model $\mathbb{M} = (\mathbb{F}, \mathrm{val})$ iff

$$\mathbb{M} \vDash \varphi_1, \ldots \mathbb{M} \vDash \varphi_n \quad \text{yields} \quad \mathbb{M} \vDash \delta.$$

A rule is **valid** in a frame $\mathbb{F}$ if it is valid in every $\mathbb{F}$-based model.

Definition

Let $\mathbb{F}$ be a Kripke frame (for modal logics) or a poset (for INT).
For $U \subseteq F$, the **generated subframe** $\mathbb{F}_X$ is the restriction of $\mathbb{F}$ to

$$\uparrow U = \{v \in F : \exists u \in U \text{ such that } u\, R^*\, v\},$$

where R^* is the reflexive-transitive closure of R.
For INT this means $\uparrow U = \{v \in F : \exists u \in F,\ u \leqslant v\}$.

A **generated submodel** $(\uparrow U, \mathrm{val}^{\uparrow U})$ is obtained by restricting the valuation:

$$\mathrm{val}^{\uparrow U}(\varphi) = \mathrm{val}(\varphi) \cap \uparrow U.$$

Fact

φ fail in a model $\mathbb{M}$ iff it fails in some of its **principal** generated submodel $\uparrow u$.

P-morphisms

Definition

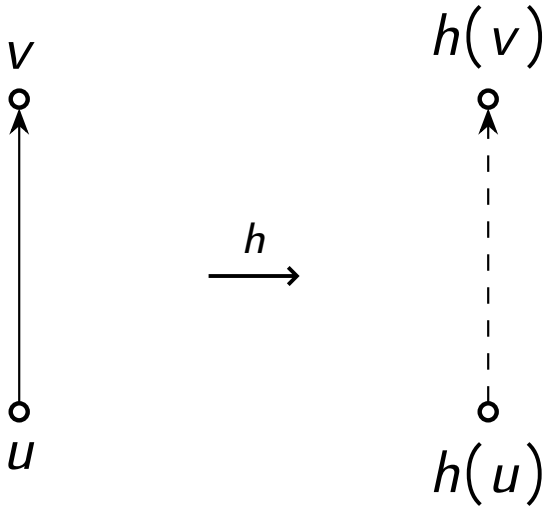
Let $\mathbb{F}$ and $\mathbb{G}$ be Kripke frames (models). A mapping $h\colon F \to G$ is called a **p-morphism** (pseudo epimorphism) provided that

(HP) for all $u, v \in F$, if $u\,R\,v$ then $h(u)\,R\,h(v)$ (*the homomorphism property*);

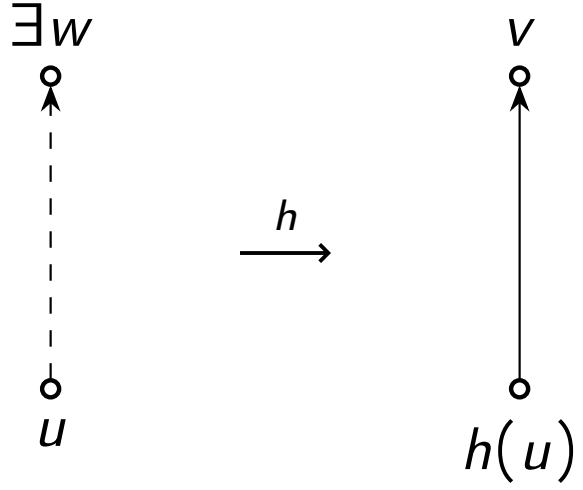
(BP) for all $u \in F$ and $v \in F$, if $h(u)\,R\,v$ then there exists $w \in F$ such that $u\,R\,w$ and $h(w) = v$ (*the backward property*),

(HaP) for all $u \in F$ and $x \in X$, $u \Vdash x$ iff $h(u) \Vdash x$ (*the harmony property*).

P-morphisms

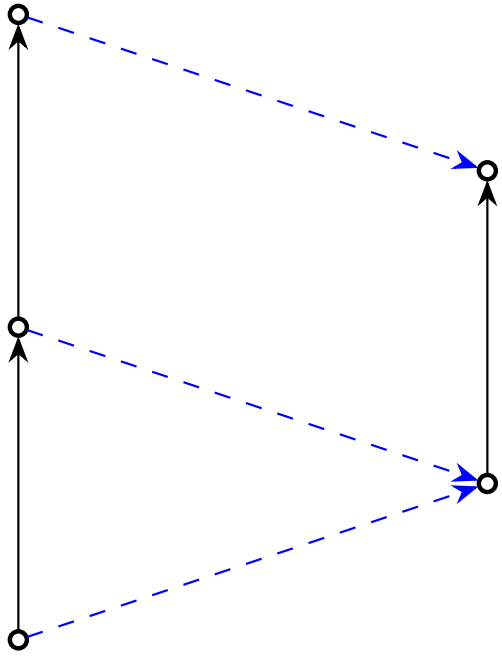


(HP)

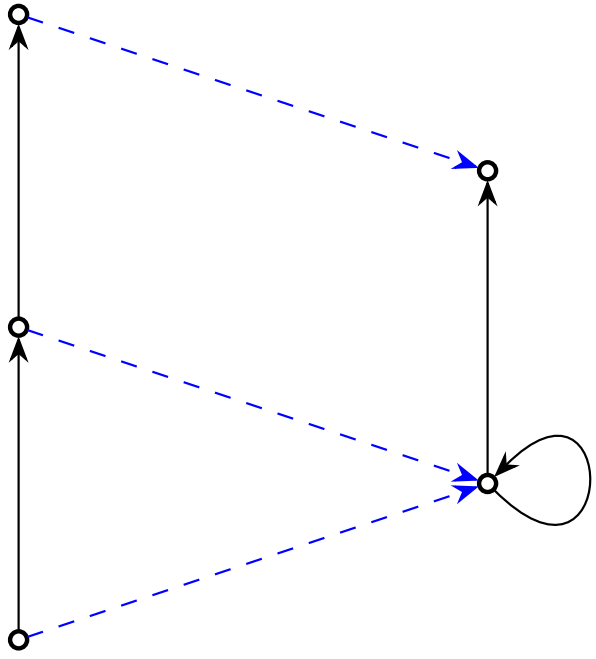


(BP)

P-morphisms



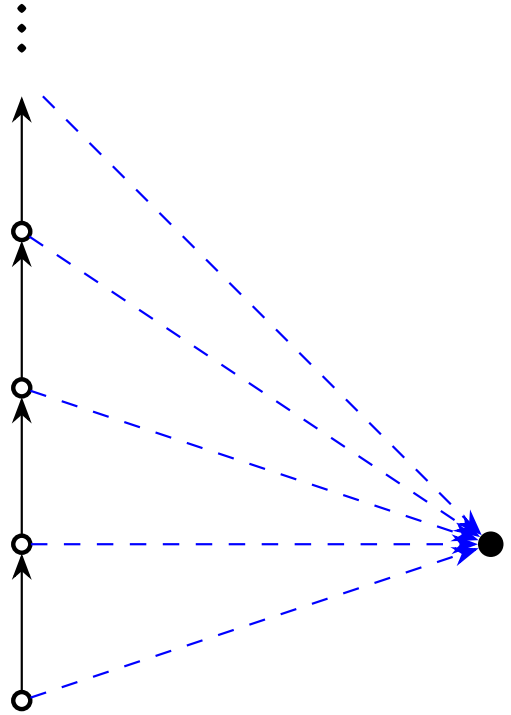
(HP) fails



(HP) holds

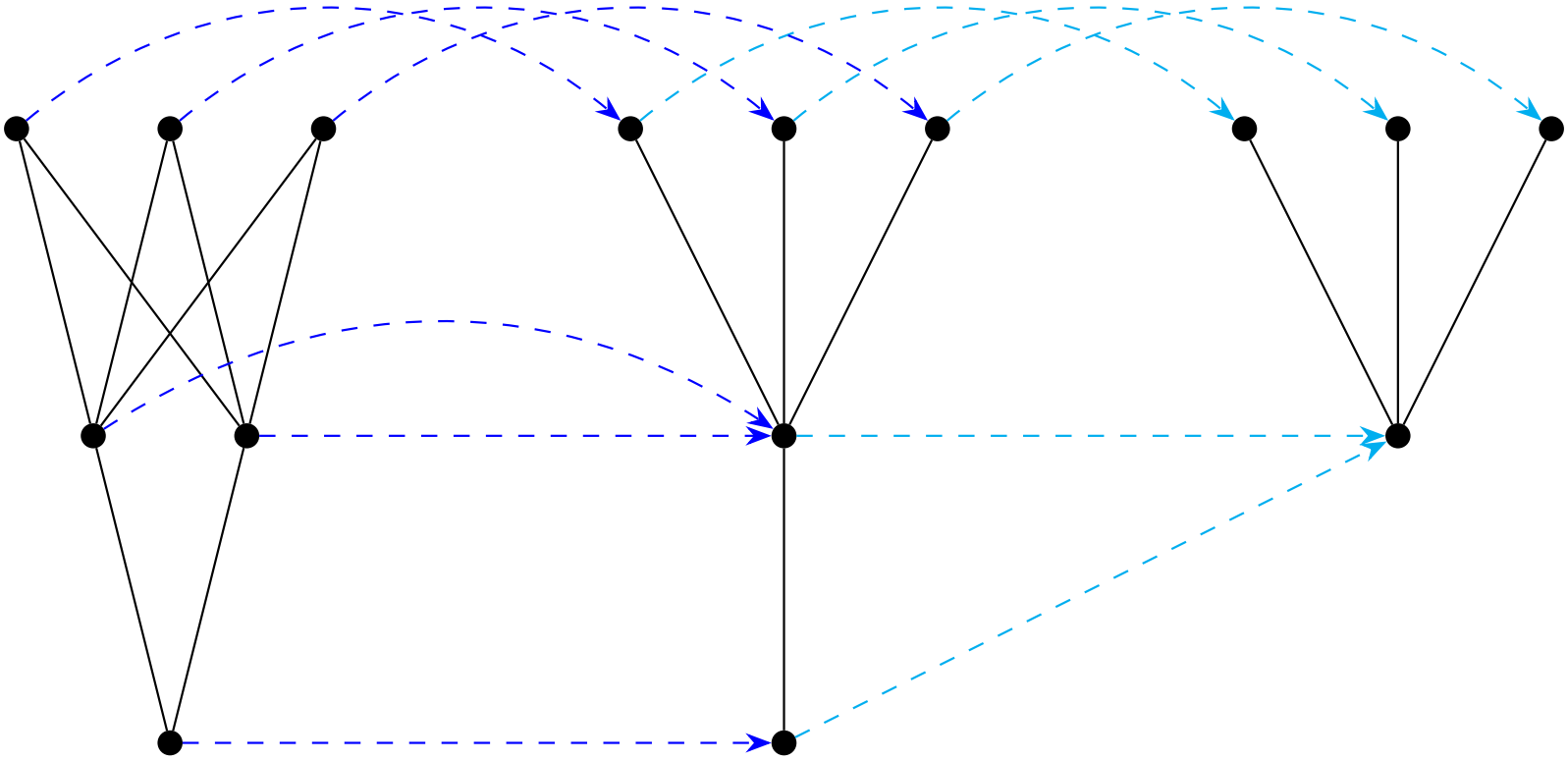
P-morphisms

(BP) fails



(BP)holds

P-morphisms for INT



β-reduction α-reduction

Fact

Let $h\colon \mathbb{M} \to \mathbb{N}$ be a p-morphism, u be a vertex in $\mathbb{M}$ and φ be a formula. Then

$$u \Vdash \varphi \quad \text{iff} \quad h(u) \Vdash \varphi.$$

Definition

An algebra in a language $\mathcal{L}$ is a non-empty set with a collections of operations that correspond to connectives in $\mathcal{L}$.

Classical examples

- Semigroups;
- (Abelian) groups;
- semilattices;
- (Bounded, distributive) lattices;
- Rings and fields;
- modules and vector spaces.

Definition

An algebra in a language $\mathcal{L}$ is a non-empty set with a collections of operations that correspond to connectives in $\mathcal{L}$.

Examples in logic

- (for CL) Boolean algebras;
- (for K) Modal algebras, i.e., boolean algebras equipped with an operation $\Box$ satisfying $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$;
- (for S4) Closure (aka interior) algebras, i.e., modal algebras satisfying $\Box \Box a = \Box a \leqslant a$;
- (for S5) Monadic algras, i.e., closure algebras satisfying $\Box(a \vee \Box b) = \Box a \vee \Box b$;
- (for INT) Heyting algebras, i.e., bounded distribute lattices equipped with $\rightarrow$ operation such that

$$a \wedge b \leqslant c \quad \text{iff} \quad a \leqslant b \rightarrow c.$$

Definition

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be algebras of the same language.
A **homomorphism** $h : \mathbf{A} \to \mathbf{B}$ is a map $h : A \to B$ such that for every operation op of arity n:

$$h(op(a_1, \ldots, a_n)) = op(h(a_1), \ldots, h(a_n)).$$

A **subalgebra** $\boldsymbol{A}$ of an algebra $\boldsymbol{B}$ is an algebra with $A \subseteq B$ and which operations are restrictions of operations in $\boldsymbol{B}$.
The **product** $\prod_{i \in I} \mathbf{A}_i$ has carrier $\prod_{i \in I} A_i$ and operations defined pointwise:

$$op((a_1^i)_{i \in I}, \ldots, (a_n^i)_{i \in I}) = (op(a_1^i, \ldots, a_n^i))_{i \in I}.$$

Definition

A **congruence** on an algebra $\mathbf{A}$ is an equivalence relation $\theta \subseteq A^2$ compatible with all operations: for every n-ary operation op,

$$a_i \theta b_i \text{ for all } i \leqslant n \quad \text{yields} \quad op(a_1, \ldots, a_n) \theta op(b_1, \ldots, b_n).$$

The **quotient algebra** $\mathbf{A}/\theta$ has carrier $A/\theta = \{a/\theta : a \in A\}$ with operations defined by

$$op(a_1/\theta, \ldots, a_n/\theta) = op(a_1, \ldots, a_n)/\theta.$$

Compatibility condition is just for the correctness of the definition of quotient.

Isomorphism Theorem

If $h : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism, then $\mathbf{A}/\ker(h) \approx \boldsymbol{B}$, where $\ker(h) = \{(a, b) : h(a) = h(b)\}$.

Subdirect product

Definition

A **subdirect product** if a subalgebra of a product (with a non-empty indexing set) with all projections surjective.

Fact

Up to isomorphisms, all subdirect product look like this:
Let θ_i be a collection of congruences on $\boldsymbol{A}$ such that $\bigcap \theta_i = id_A$.
Let $i \colon \boldsymbol{A} \to \prod_i \boldsymbol{A}/\theta_i$ be

$$i(a) = (a/\theta_i)_{i \in I}.$$

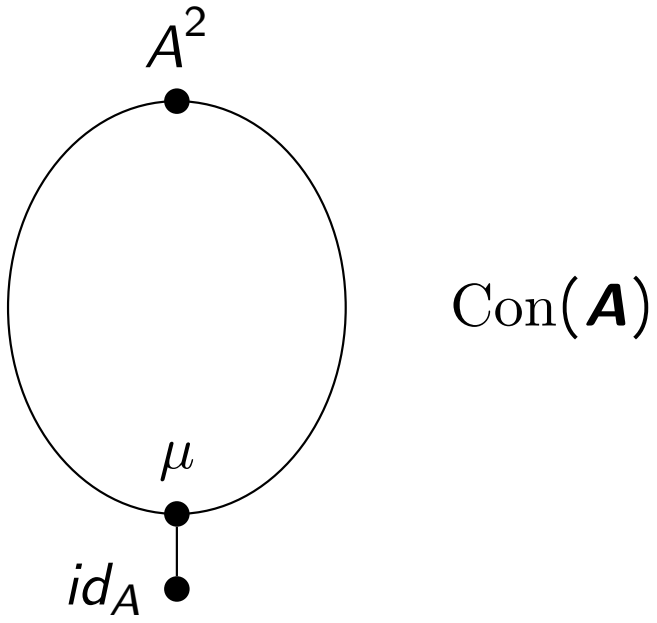
Then i is an embedding and $i(\boldsymbol{A})$ is a subdirect product of $\boldsymbol{A}/\theta_i$.

Definition

An algebra is **subdirectly irreducible** if, for every its subdirect product representation, at least one projection is an isomorphism.

Fact

An algebra $\boldsymbol{A}$ is subdirectly irreducible iff it has a unique minimal (above id_A) congruence μ. The congruence μ is called **monolith**.



Examples

- Every simple (with no congruences apart of A^2 and id_A) algebra;
- The only subdirectly irreducible boolean algebra is $\mathbf{2}$;
- The only subdirectly irreducible distributive lattice is $\mathbf{2}$;
- A Heyting algebra is subdirectly irreducible iff it has a unique maximal element;
- A closure algebra is subdirectly irreducible iff it has a unique maximal open ($\Box a = a$) element.

Theorem (Birkhoff)

Every algebra is a subdirect product of subdirectly irreducible algebras.

For an intuitionistic frame $\mathbb{F}$, let $\mathbb{F}^*$ be the Heyting algebra of all its upsets (equiv, generated subframes) with set operations $0 = \varnothing$, $1 = F$, $\wedge = \cap, \vee = \cup$, and

$$a \to b = \{u : \forall v \text{ if } u \leqslant v \text{ and } v \in a \text{ then } v \in b\}.$$

For an modal frame $\mathbb{F}$, let $\mathbb{F}^*$ be the modal algebra of of its subsets with set operations $0 = \varnothing$, $1 = F$, $\wedge = \cap, \vee = \cup$, $\neg = F\setminus$, and

$$\Box a = \{u : \forall v \text{ if } u\, R\, v \text{ then } v \in a\}.$$

For a (modal or intuitionistic) p-morphism $f : \mathbb{F} \to \mathbb{G}$ let $f^* : \mathbb{G}^* \to \mathbb{F}^*$ be a homomorphism given by

$$f^*(b) = f^{-1}(b).$$

Theorem

Every finite Heyting or modal algebra is isomorphic to some $\mathbb{F}^*$.
Also, every homomorphism between Heyting or modal algebras is, up to isomorism, f^*.
More formally, $(\ )*$ is a dual isomorphism from the category of intuitionistic/modal frames with p-morphisms onto the category of finite Heyting/modal algebras with homomorphisms.
The inverse functor is denoted by $(\ )_*$.

Properties

In this dualities ww have correspondences

frame	algebra
p-morphism	homomorpsism
generated submodel	surjective homomorphism
surjective p-morphism	subalgebra
disjoint union	product
rooted frame	subdirectly irreducible algebra

Remark

There are, of course, finer dualities that works also for infinite objects.

However, we will try not to use them in this course.